Multiple membrane cavity optomechanics

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We investigate theoretically the extension of cavity optomechanics to multiple membrane systems. We describe such a system in terms of the coupling of the collective normal modes of the membrane array to the light fields. We show these modes can be optically addressed individually and be cooled, trapped and characterized, e.g. via quantum nondemolition measurements. Analogies between this system and a linear chain of trapped ions or dipolar molecules imply the possibility of related applications in the quantum regime.

Cavity optomechanics is an emerging field at the boundary between quantum optics and nanoscience. Resulting in part from experimental innovations at the mesoscopic scale, optomechanical systems – mechanical systems that can be manipulated by light – have recently generated much experimental and theoretical interest [1, 2, 3, 4, 5, 6]. They offer the prospect of realizing quantum effects at a macroscopic scale [7], of supplying novel quantum sensors for applications ranging from single molecule detection [8] to gravitational wave interferometry [6, 9], for the quantum control of atomic, molecular and optical systems [10], and for possible new quantum information processing devices [11].

As cavity optomechanics begins to mature as a field, we recognize that scalability is an important aspect of any technology. In particular, it is immediately relevant to possible uses in information processing. This has been acknowledged in proposals for constructing quantum computers using trapped ions, cavity quantum electrodynamics, neutral-atomic lattices, nuclear magnetic resonance, spintronics, dipolar molecules, etc., see Ref. [12] and references therein. It is therefore important to investigate the scaling of current cavity-based techniques to a larger number of optomechanical elements.

In this Letter we consider the optomechanical cooling and trapping of a small array of partially transparent dielectric membranes inside a high-finesse cavity driven by laser radiation (Fig. 1). Presenting first the case of two moving membranes, $N = 2$, where explicit analytical results are readily obtainable, we show that it is described most conveniently in terms of the optical cooling, trapping and measurement of the *normal modes* of the linear chain formed by the membranes. This demonstration depends crucially on the fact that the optical spectrum of the cavity is symmetric in the symmetric normal modes of the membrane array, while it does not have well-defined symmetries with respect to the motion of the individual membranes. We then extrapolate these considerations to the case of N membranes in a cavity, and draw an analogy to chains of trapped ions [13, 14] and of dipolar molecules [15]. We conjecture that this analogy may
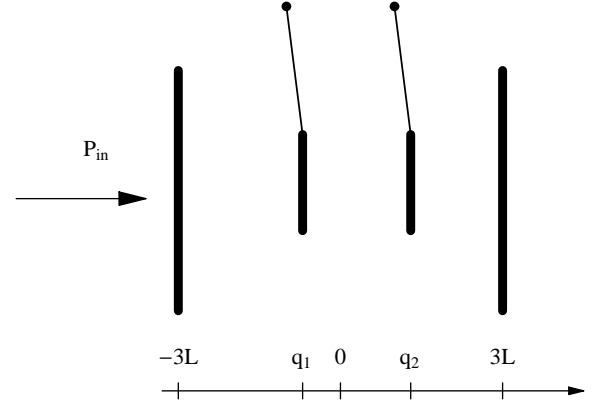

FIG. 1: A high finesse optical cavity with two mirrors fixed at $\pm 3L$, and two dielectric membranes centered at $q_1 \sim -L$ and $q_2 \sim L$ respectively. P_{in} signifies the power in the laser light that may be used to drive the cavity.

have implications for information processing if, as expected, the membranes can be experimentally placed in the quantum regime.

Our immediate motivation derives from a pioneering experiment that demonstrated that a single membrane can be optomechanically cooled when placed inside a high finesse optical resonator [16]. This is the first experiment to show that the two technologically challenging requirements of high optical and mechanical quality can be allocated separately to the end-mirrors and the dielectric membrane respectively, distinct from the standard two-mirror cavity configuration [1]. This work also pointed out that in addition to the usual linear coupling of radiation to the displacement q of the membrane that facilitates back-action cooling, a coupling proportional to q^2 is also possible. If the membrane oscillation frequency is larger than the optical linewidth of the cavity [17, 18, 19], this quadratic interaction enables quantum non-demolition measurements of the energy of the vibrating membrane. Linear couplings do not allow such measurements [20].

The present work draws on our previous analysis of

the case of a single membrane [21], which yields an approximate microscopic Hamiltonian that describes the optomechanics of the system. We show that this formalism can in principle be generalized to N membranes, with the generic features extracted simply from considering the case $N = 2$ (Fig. 1). More detailed calculations will be presented elsewhere [22].

Our starting point is a cavity with two fixed and perfectly reflecting end mirrors and two identical vibrating dielectric membranes, each of reflectivity R, mass m and mechanical frequency ω_m, see Fig. 1. We model the system with the quantum mechanical Hamiltonian [21]

$$H = \sum_j^2 \left(\frac{p_j^2}{2m} + \frac{1}{2} m \omega_m^2 q_j^2 \right) + \sum_i^3 \hbar \omega_i(q_j) a_i^\dagger a_i, \quad (1)$$

where p_j and q_j, $j = 1, 2$ denote the momentum and position of the membranes and ω_i, $i = 1, 2, 3$, are the resonance frequencies of three near-degenerate optical modes with creation and destruction operators a_i and $a_i^\dagger$ respectively. The commutation rules obeyed by these operators are $[q_j, p_l] = i\hbar \delta_{jl}$ and $[a_i, a_n^\dagger] = \delta_{in}$. We discuss shortly why three optical modes are sufficient for a minimal description of the system.

The first term in Eq. (1) is the mechanical energy of the oscillating membranes, and the second term the energy of optical modes of the full resonator of length $6L$. The dependence of the cavity mode frequencies ω_i on the positions q_j of the membranes is central to the description of the system, since it determines the optomechanical couplings [21]. The ω_i are obtained by solving the classical Maxwell equations for the full resonator. We assume that the moving membranes are much thinner than an optical wavelength and model them by spatial delta functions [21]. For $R = 1$, the resonator consists simply of three uncoupled cavities whose eigenfrequencies in the absence of membrane motion ($q_{1,2} = \mp L$) are threefold degenerate and are given by

$$\omega_n = \frac{n\pi c}{2L}. \quad (2)$$

Here n is a positive integer, c is the velocity of light, and $2L$ is the length of each sub-cavity. This is the reason we included only three modes in Eq. (1); the spectrum may generally be grouped in such triplets.

For $R \neq 1$, the three resonators are coupled, this coupling lifting the degeneracy of the frequencies ω_n. Solving Maxwell's equations with the appropriate boundary conditions at the membranes, and neglecting light absorption in these membranes, we find that the allowed wave-numbers of the complete resonator can be expressed in terms of the relative coordinate $q = q_1 - q_2$ and the center-of-mass (COM) coordinate $Q = (q_1 + q_2)/2$ of the membranes via the trigonometric equation

$$\sin 2(\theta + 3kL) + \sin^2 \theta \sin 2k (3L - q)$$
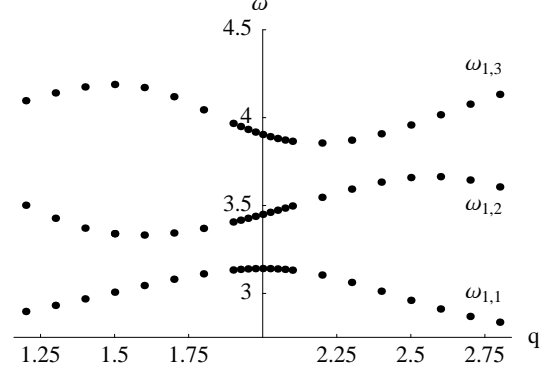$$= 2 \sin \theta \cos (\theta + kq) \cos 2kQ, \quad (3)$$

FIG. 2: A portion of the optical frequency spectrum for the two-membrane cavity. A closely spaced triplet of frequencies $\omega_{1,i}(i = 1, 2, 3)$ is shown, using a higher resolution near the mode equilibrium point $q_0 = 2$. For $Q \neq 0$ the optical mode frequencies are similarly modulated in a direction orthogonal to q. The cavity parameters used in the plot are $R = 0.5, L = 1$.

where $\sin \theta = \sqrt{R}$.

Equation (3) is invariant under the transformation $Q \rightarrow -Q$. Physically this means that the frequency spectrum is symmetric in the COM motion. We show below that this symmetry can be exploited to obtain a description of the system purely in terms of the 'phonon' modes associated with the relative and COM coordinates q and Q separately, that is, excluding any effects resulting from the coupling between these coordinates. This is in contrast to an equivalent description in terms of the individual oscillator coordinates q_1 and q_2, which typically retains coupling terms.

Equation (3) can be solved numerically for a given set of system parameters. Figure 2 shows a triplet of optical modes $\omega_{n,i}(q, Q)$ in the vicinity of the equilibrium point of relative motion $q_0 = 2$ and for $Q = 0$. The optical frequencies are strongly modulated along q, a situation familiar from the case of a single membrane, $N = 1$. These modulations arise from the avoided crossings associated with the lifting of the degeneracy of the frequency triplet due to the coupling of the sub-cavities [22]. Similar periodic modulations appear in $\omega_{n,i}(q, Q)$ for fixed q and Q varying.

Considerable insight can be gained from approximate analytical solutions of Eq. (3) [22]. Perturbation theory in the small parameters $(q - q_0, Q - Q_0) \ll 2\pi c/\omega_n$ yields for example the triplet ($i = 1, 2, 3$) of optical frequencies in the vicinity of any q_0 and Q_0

$$\begin{aligned} \omega_{n,i}(q, Q) = {} & \Delta_{n,i} + B_{n,i}(q - q_0) + B'_{n,i}(Q - Q_0) \\ & + M_{n,i}(q - q_0)^2 + M'_{n,i}(Q - Q_0)^2 \quad (4) \\ & + P_{n,i}(q - q_0)(Q - Q_0) + \dots. \end{aligned}$$

The various terms in this equation fully describe the

basic optomechanical properties of the system. Specifically, $\Delta_{n,i}$ are the frequency shifts due to the sub-resonators coupling in the absence of membrane motion. $B_{n,i}$ and $B'_{n,i}$ determine the strength of the linear optomechanical couplings associated with the "breathing" and COM modes of motion, respectively, producing a back-action of the mirror motion on the light field that can be exploited in mirror cooling. $M_{n,i}$ and M'_i govern the quadratic couplings and can lead to quantum non-demolition (QND) energy measurements [16]). Finally $P_{n,i}$ quantifies the coupling between the relative and COM modes and is responsible for normal mode decoherence as well as down-conversion [23]. We discuss later in this Letter circumstances under which that coupling can be cancelled.

Consider for concreteness the case $q_0 = 2L, Q_0 = 0$. In this case we have

$$\Delta_{n,1} = \frac{n\pi c}{2L},$$

$$\Delta_{n,2} = \frac{c}{2L}\left[n\pi + \sin^{-1}\left(\frac{\sqrt{3+\sin^2\theta}}{2}\right) - \theta\right], \qquad (5)$$

$$\Delta_{n,3} = \frac{c}{2L}\left[(n+1)\pi - \sin^{-1}\left(\frac{\sqrt{3+\sin^2\theta}}{2}\right) - \theta\right],$$

and

$$B_{n,i} =$$
$$\frac{\xi_{n,i}\sin(\theta_{n,i})\sin(2\theta)}{3\cos(\theta_{n,i})\sin^2(\theta) + 3\cos(2\theta + 3\theta_{n,i}) + \sin(2\theta)\sin(\theta_{n,i})}, \qquad (6)$$

where $\theta_{n,i} = 2\Delta_{n,i}L/c$ and $\xi_{n,i} = \theta_{n,i}/(2L)^2$.

In that case, both $B_{n,2}$ and $B_{n,3}$, which are the slopes of the two upper curves in Fig. 2, are different from zero. The linear spatial dependence of the optical frequencies on q can therefore be used for back-action cooling and trapping of the "breathing" mode of the pair of membranes. In contrast, $B_{n,1} = 0$, and the corresponding resonator mode cannot be used for that purpose. However it can be used for a non-demolition measurement of the q mode, as it provides a purely quadratic coupling

$$M_{n,1} = -\frac{1}{6}\tau\xi_{n,1}^2\tan\theta \neq 0, \qquad (7)$$

where $\tau = 4L/c$ is the round trip time for each sub-cavity. Hence both linear and quadratic optomechanical couplings are accessible at the same point $q_0 = 2$ using modes from the same frequency triplet. This is unlike the situation for a single membrane, see Refs. [16, 21].

The analysis for the mode Q is similar to the case of a single membrane since both describe the effects of COM motion [21]. The choice of $Q_0 = 0$ implies the absence of the terms linear in Q in the expansion of Eq. (4), a direct consequence of the fact that Eq. (3) is symmetric

about $Q_0 = 0$. Hence $B'_{n,i} = 0$. The quadratic coupling is however non-zero; for reasons of space we will provide an expression for it elsewhere [22]. On the other hand, when the membranes are positioned so that $Q_0 \neq 0$ then non-vanishing terms linear in Q arise in Eq. (4) and optomechanical center-of-mass cooling becomes possible. Further analysis of this case will be presented elsewhere [22].

We finally turn to the last term in Eq. (4). As already mentioned, it describes the coupling between the breathing and COM modes of motion of the membranes. In perturbation theory this bilinear term is smaller than the linear terms, hence it does not effect significantly the back-action properties used in cooling. Furthermore, for $Q_0 = 0$ we must have that $P_{n,i} = 0$ since Eq. (3) is even in Q and hence the expansion cannot contain terms that are odd in Q. To lowest order, the coupling between these modes of motion is then proportional to $(q-q_o)Q^2$ and is negligible. In that case, the breathing and COM modes remain to an excellent approximation the 'true' normal modes of the full two-membrane optomechanical problem. This is a central result of this paper. In physical terms, this implies that with an appropriate combination of optical frequencies, it is possible to independently cool, trap and characterize the breathing and COM modes of the 2-membrane array.

We now discuss in broad terms the extension of these considerations to an array of $N > 2$ membranes. The addition of a membrane positioned at a node of the intracavity field does not change the optical finesse measurably ; if the mirror is placed away from a frequency extremum, the finesse is multiplied by $R \sim 0.99$ [16, 21]. Thus a scaling up to $N \sim 10$ membranes should be practical.

The boundary conditions of the optical resonator yield a trigonometric equation similar to Eq. (3) independently of the number N of membranes. It is always possible to express this equation in terms of the N normal modes Q_j of the array, since they are linear combinations of the individual oscillator coordinates q_j. Generally the resulting optical spectrum will be periodic in every mechanical mode, guaranteeing the existence of ranges where the dependence of the $N + 1$ frequencies $\omega_{n,i}(\{Q_j\}), i = 1, \ldots N + 1$, exhibits extrema, and other regions of linear dependence. This in turn implies the possibility of cooling, trapping and performing QND measurements of the energy of all collective modes. The optomechanical parameters coupling any optical mode to any mechanical mode can be extracted either analytically, from expressions such as Eqs. (4-7), or numerically. The cooling and trapping effects due to back-action , being linear in the Q_j, are always independent of the coupling between the phonon modes.

Most importantly, we expect the optical spectrum to be symmetric in every symmetric normal mode of the array. We have shown this explicitly for the COM mode in

the cases $N = 1$ [21] and $N = 2$ (this work). There are $N/2$ symmetric modes for N even, and $N/2 + 1$ for N odd [24]. For $N = 3$ for example, with individual oscillator coordinates $q_j, (j = 1, 2, 3)$ we expect the optical spectrum to be symmetric in the COM mode $Q_1 = (q_1 + q_2 + q_3)/3$ as well as in the 'scissors' mode $Q_2 = (q_1 - 2q_2 + q_3)/6$, but not in the stretch mode $Q_3 = (q_3 - q_1)/2$ [22]. However the amount of symmetry available is adequate to cancel all mode-mode coupling terms by appropriate positioning of the membranes. For $N \geq 4$, some mode coupling terms will have to be included in the description of the optomechanics. For N even, for example, $\binom{N/2}{2} = N(N-2)/8$ coupling terms out of a total possible $\binom{N}{2} = N(N-1)/2$ will remain in the expression for each frequency.

The full behavior of the system may be modelled by a Hamiltonian analogous to Eq. (1), combined with a quantum treatment of the noise associated with the input fields and with dissipation, see e.g. Ref. [18]. Experimentally the noise spectrum of any collective modes can be obtained in principle by modulating the light field at the specific mode frequency [3].

Our description of the membrane array is reminiscent of a chain of trapped ions [13, 14, 23, 24], with however significant differences. First, in the present case the coupling between mirrors is due to radiation pressure rather than the Coulomb interaction, and is therefore switchable. As such, the situation is perhaps more akin to the case of dipolar molecules, where the coupling is due to the dipole-dipole interaction and can also be tuned. Also, membranes and mirrors do not possess internal degrees of freedom, at least at the level of the present description. Still, the analogy to the physics of trapped ions or dipolar molecules on lattices raises the possibility of using multi-membrane systems as information processing devices. Each membrane can be addressed individually through its suspension, and additional degrees of freedom such as rotation could possibly be added and exploited [25]. Issues of noise and decoherence will of course be extremely challenging, but some work along these lines is already being carried out in efforts to cool such systems to their vibrational ground state [17, 18]. In addition, as is the case for trapped ions the independent control of the collective normal modes of the array will become technically increasingly challenging with increasing N.

In conclusion we have considered the extension of cavity optomechanics to multi-membrane systems, and found that the normal modes of the system can cooled, trapped and measured. The similarity of this system to chains of trapped ions or dipoles leads us to conjecture that it may find possible future application as an information-processing device operated in the quantum regime.

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